Linear quadratic Gaussian control for adaptive optics systems using a numerical atmospheric turbulence model

Jean-Pierre Folcher[1a] and Marcel Carbillet[1]

UMR 7293 J.-L. Lagrange
Université de Nice Sophia-Antipolis/CNRS/Observatoire de la Côte d'Azur
Parc Valrose, 06108 Nice cedex 2, France

Abstract. This paper presents a linear quadratic Gaussian (LQG) command based design approach for the control of an astronomical adaptive optics system. In this framework, the control uses the state-feedback of the atmospheric distortion wavefront estimate. Such estimate is obtained from a Kalman filter which incorporates a model of the atmospheric distortion wavefront. During on-sky observations, strength of the atmospheric turbulence and wind velocity of each turbulent layer can change rapidly, degrading the atmospheric distortion wavefront estimate. We derive a numerical procedure in order to obtain a model of the atmospheric distortion wavefront which guarantees satisfactory disturbance rejection performance in despite of turbulence variations. Numerical experiments using the `Software Package CAOS` have been conducted to demonstrate the robustness of the proposed approach.

1 Introduction

Adaptive optics (AO) systems are used in order to counter the effects of atmospheric turbulence on the phase of the incoming light, in particular when imaging astronomical objects using ground-based telescopes. In a standard AO system, the atmospheric wavefront is reflected on a deformable mirror (DM) which deforms its surface in order to obtain a residual wavefront as close as possible to a theoretical plane wavefront, while a wavefront sensor (WFS) analyzes the distortion of the residual wavefront by integrating the measures made over the frame of a CCD camera. Based on this measured output, a controller generates the voltages applied to each DM actuators by a zero-order hold (ZOH) which adjusts in real time the DM shape. Such a system can be viewed clearly as a feedback system and currently many control methods are effective for an increasing number of AO systems [2,3]. For an overview of AO, the reader may consult the book edited by Roddier in 1999 [1].

Generally an AO system is not able to react instantaneously to the atmospheric wavefront disturbances: the DM response is delayed and the residual wavefront presents a temporal error. This delayed response is caused by the cumulative delay of the WFS (exposure time and read-out of the CCD camera), of the controller (computational delay), and of the DM dynamics which limits the bandwidth of the loop transfer function. Consequently, a fast atmospheric wavefront dynamics relative to the cumulative delay will induce a significant degradation of the image quality. Therefore the AO system performance depends on the ability of the control to have a reasonable complexity (to limit the computational delay) while taking into account the temporal evolution of the atmospheric wavefront. Minimum-variance control which consists in minimizing the mean-square residual wavefront error appears as a convenient approach to tackle this problem (see [4–6]). This approach can be formulated as a linear quadratic Gaussian (LGQ) control problem, and the solution consists in the optimal state-feedback control of the DM and the optimal estimation of the atmospheric wavefront. In section 3 of the present paper we consider the LQG design method [7] which explicitly considers the disturbance rejection performance and facilitates the numerical resolution (low order estimation Algebraic Riccati Equation).

During on-sky observations, strength of the atmospheric turbulence and wind velocity of each turbulent layer can change rapidly. Thus a compelling issue is the determination of the model which can fully capture the temporal evolution of the atmospheric wavefront. The choice of auto-regressive

[a] Jean-Pierre.FOLCHER@unice.fr

(AR) models, as in paper [4], ensures a good trade-off between complexity and precision. Different atmospheric model identification approaches (cross-correlation structure fitting[4], subspace identification [8]) have been developed, and can be viewed as a straightforward way for the computation of the LQG controller parameters. Consequently, the resulting LQG controller is optimal in the mean-square sense for the given atmospheric model (which partially represents the atmospheric turbulence system). Unfortunately, in this context the LQG design does not guarantee performance when the atmospheric system is subject to uncertainties or variations. In order to provide robust mean-square performance, we consider the performance criterion in the frequency domain as in paper [6]. In this context, the AR atmospheric model is considered as a weighting function. LQG design can be interpreted as a way to shape the sensitivity transfer function and the disturbance rejection transfer function [7] with respect to the AR weighting function. A constructive numerical method of the AR weighting function is developped to guarantee robust performance of the adaptive optics feedback system.

2 Adaptive optics control problem

In this paper we assume that the wavefront over the aperture can be represented by the modal expansion of the wavefront on the Zernike basis of dimension n_b. At each time instant k, the atmospheric wavefront, the mirror shape correction and the residual wavefront error are described by the finite-dimensional vectors $w_a \in \mathbf{R}^{n_b}$, $w_m \in \mathbf{R}^{n_b}$ and $w_r \in \mathbf{R}^{n_b}$. The closed loop AO system is depicted in figure 1. The DM is computer-controlled using command input $u \in \mathbf{R}^{n_u}$ and the WFS produces a discrete-time measurement $y \in \mathbf{R}^{n_y}$. The behavior of the DM and the WFS is determined by the transfer function G_m, and G_w. Furthermore, the signal $n_w \in \mathbf{R}^{n_y}$ is an additive perturbation input of the WFS measurement y.

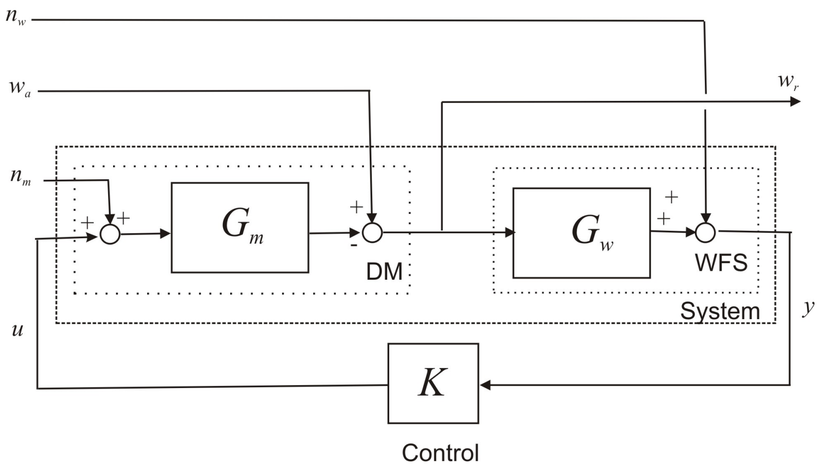

Fig. 1. AO discrete-time system block-diagram.

The main control objective (disturbance rejection) is to minimize the “size” of the residual wavefront w_r for a given set of the residual wavefront w_a and for a given set of measurement perturbation n_w. In the z-domain, we can write

$$\mathcal{Z}\{w_r\} = (I + L(z))^{-1}\,\mathcal{Z}\{w_a\} + (I + L(z))^{-1}\,G_m(z)K(z)\mathcal{Z}\{n_w\}\ , \tag{1}$$

where $\mathcal{Z}\{.\}$ stands for the z-transform of the considered signal and where $L(z) = G_m(z)K(z)G_w(z)$ is the loop transfer function. If we refer to equation (1), we call the sensitivity transfer function,

$$T_{11}(z) = (I + L(z))^{-1}\ ,$$

and the disturbance rejection transfer function

$$T_{12}(z) = (I + L(z))^{-1} G_m(z)K(z) .$$

The disturbance rejection performance is entirely determined by these two transfer functions. At this step no assumption is made for the type of controller (integral, LQG, ...) for the set of the perturbation inputs w_a and n_w. The performance criterion, the "size" of the residual wavefront w_r is also not defined. A first approach involves frequency response analysis of the closed loop transfer function which provides some crucial information about the system performances (stability, disturbance rejection, command input peak value), see for instance the book [9]. In a more complete framework some properties of the AO loop (including disturbance rejection performance, stability margins) have been analyzed in the paper [7].

In the present paper we decide to evaluated the 'size" of the residual wavefront w_r in terms of variance (mean-square error) which has a straightforward link with the imaging performance of AO systems. We assume that signals w_r, w_a, n_w are stationary and independent stochastic signals. Thus, in the frequency domain, the variance $\mathbf{E}\left[w_r(k)^T w_r(k)\right]$ can be written as

$$\begin{aligned}\mathbf{E}\left[w_r(k)^T w_r(k)\right] &= \tfrac{T}{2\pi}\int_0^{\frac{2\pi}{T}} \mathbf{Tr}\left(T_{11}(e^{j\omega T})S_{w_a}(e^{j\omega T})T_{11}(e^{-j\omega T})^T\right)d\omega \\ &+ \tfrac{T}{2\pi}\int_0^{\frac{2\pi}{T}} \mathbf{Tr}\left(T_{12}(e^{j\omega T})S_{n_w}(e^{j\omega T})T_{12}(e^{-j\omega T})^T\right)d\omega ,\end{aligned} \qquad (2)$$

where S_{w_a} and S_{n_w} are the power spectral densities of the input signals w_a and n_w. The first term of the right hand side of equation (2) represents the contribution of the atmospheric wavefront and the latter the contribution of the WFS measurement noise. Minimum variance control associated to an LQG formulation minimizes the mean-square error: it is a convenient approach to tackle this A0 control problem as in [4–6]. In the next section, we will present a standard formulation of this control method. In many cases, the power spectral density S_{n_w} is taken constant. For a given power spectral densities S_{w_a}, LQG control design realizes a trade off between the contribution of the atmospheric wavefront and the contribution of the WFS measurement noise. Atmospheric wavefront can be modeled as the output of a strictly causal linear time invariant (LTI) system described by a transfer function G_a where the input is a zero-mean white noise signal with unitary covariance matrix. Power spectral densities which is defined as

$$S_{w_a}(w) = G_a(e^{j\omega T})G_a(e^{-j\omega T})^T,$$

can be seen as a weighting function in equation (2) and indicates the frequency range where the frequency responses $T_{11}(e^{j\omega T})$ has to be small.

The compelling issue is the choice of the atmospheric wavefront's model G_a. Different atmospheric AR model identification approaches (cross-correlation structure fitting[4] , subspace identification [8]) have been proposed. Unfortunately, in this context, only a nominal atmospheric model is considered and LQG design does not guarantee performance when the atmospheric system is subject to uncertainty or variations. In order to ensure mean-square performance despite the variations of atmospheric turbulence, we proposed a slightly different numerical approach to obtain a second order diagonal AR model (to take into account the oscillating behavior of time evolution)

$$A_0 w_a(k) + A_1 w_a(k-1) + A_2 w_a(k-2) = n_a(k-1) , \qquad (3)$$

where input $n_a \in \mathbf{R}^{n_b}$ is a zero-mean white stochastic process with unitary covariance matrix. Several temporal evolutions of a 3-layers turbulent atmosphere were generated using the `Software Package CAOS` [10]. For each projected wavefront, the Burg algorithm [11] gives the stable AR system's parameters (A_0, A_1, A_2) that minimize the sum of the squares of the forward and backward prediction errors. In the frequency domain, we have considered the set of model's response and built a worst case model and a nominal model. These two atmospheric models are used to design LQG controllers. A posteriori MIMO frequency analysis and time simulation are achieved to select the controller which ensures acceptable performance for the set of atmospheric wavefront trajectories. Note that this numerical method permits to take into account for the controller design some turbulent atmospheres where the physical parameters (Fried parameter, outer scale, wind velocities) are different.

3 Linear quadratic Gaussian framework

The Strehl ratio is a convenient measure for the imaging performance of AO systems and is a strictly decreasing function of the residual wavefront variance. A relevant AO control objective is the minimization of the residual wavefront variance. Thus, the control problem can be expressed as finding the control law that minimizes the empirical variance of the residual wavefront, averaged over a large exposure time. As the control loop is discrete time, the equivalent discrete time performance criterion see [6] is defined as $J_r = \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \|w_r(k)\|^2$. We add a quadratic penalty on the control to take into account command input specification. We consider the performance index

$$J = \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \|w_r(k)\|^2 + \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} u(k)^T R u(k) \ , \tag{4}$$

where the residual wavefront is given by $w_r(k) = w_a(k) - w_m(k)$ and with the weighting matrix $R = R^T > 0$. We consider that the mirror's time response is negligible compared to the sampling period T. The corrected wavefront is assumed to depend linearly on the delayed command input and on a disturbance input modeling uncertainties (saturations, neglected dynamics). The relation between the command input and the mirror shape correction (including a unitary command input delay) is $w_m(k) = M_m u(k-1)$ where M_m stands for the DM influence matrix. Moreover we assume that WFS works in a linear range and provides a linear relation between the residual wavefront and includes a unitary delay (read-out and slopes computation time). The WFS measurements are corrupted by a noise, that is $y(k) = M_w w_r(k-1) + n_w(k)$ where M_w is the WFS influence matrix. To obtain the LQG control that minimizes the criterion (4), we express the augmented system (AO loop including the atmospheric model) in state space form as

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} n_a(k) \ , \\ y(k) &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} x(k) + n_w(k) \ , \end{aligned} \tag{5}$$

where the state $x_1(k) = \begin{bmatrix} w_m(k)^T & w_m(k-1)^T \end{bmatrix}^T$ represents the plant dynamics (DM & WFS) and state $x_2(k) = \begin{bmatrix} w_a(k)^T & w_a(k-1)^T \end{bmatrix}$ corresponds to the perturbation dynamics (atmospheric model). $n_a(k) \in \mathbf{R}^n$ represents the state disturbance and $n_w(k) \in \mathbf{R}^{n_y}$ is the output noise. We assume that Gaussian noise processes $w(k)$ and $v(k)$ are mutually independent, zero mean white noises with covariance $\mathbf{E}\left[n_a(k) n_a^T(l)\right] = I\delta(k-l)$ and $\mathbf{E}\left[n_w(k) n_w^T(l)\right] = V\delta(k-l)$, respectively. We have also

$$A_1 = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -A_0^{-1}A_1 & -A_0^{-1}A_2 \\ I & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} M_m \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} A_0^{-1} \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & -M_w \end{bmatrix},$$

and $C_2 = \begin{bmatrix} M_w & 0 \end{bmatrix}$. We write the performance criterion (4) under the generic equivalent form

$$J = \lim_{K\to\infty} \frac{1}{K} \mathbf{E} \left[\sum_{k=0}^{K-1} x(k)^T Q x(k) + u(k)^T R u(k) \right] \ , \tag{6}$$

with given weighting matrices $Q = \begin{bmatrix} 0 & -I & I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -I & I & 0 \end{bmatrix}$ and $R = R^T > 0$. The LQG controller (interconnection of a quadratic regulator and a linear state estimator) is described by

$$\begin{aligned} \begin{bmatrix} \hat{x}_1(k+1) \\ \hat{x}_2(k+1) \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ -L_2 C_2 & A_2 - L_2 C_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \\ u(k) &= \begin{bmatrix} -K_1 & -K_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} \ , \end{aligned} \tag{7}$$

where the optimal state feedback gains (constant matrices) are

$$K_1 = (B_1^T P_{11} B_1 + R)^{-1} B_1^T P_{11} A_1, \quad K_2 = (B_1^T P_{11} B_1 + R)^{-1} B_1^T P_{12} A_2 \ . \tag{8}$$

Matrix $P_{11} = P_{11}^T \geq 0$ is the unique positive-semidefinite solution of the discrete-time algebraic Riccati equation $P_{11} = A_1^T P_{11} A_1 - A_1^T P_{11} B_1 \left(B_1^T P_{11} B_1 + R\right)^{-1} \bar{B}_1^T P_{11} A_1 + Q_{11}$, and P_{12} is a solution of the following discrete-time Sylvester equation $P_{12} = (A_1 - B_1 K_1)^T P_{12} A_2 + Q_{12}$. The observer gain L_2 (constant matrice) is given by

$$L_2 = A_2 X_{22} C_2^T \left(C_2 X_{22} C_2^T + V\right)^{-1} , \tag{9}$$

where $X_{22} = X_{22}^T \geq 0$ is the unique positive-semidefinite solution of the algebraic Riccati equation $X_{22} = A_2 X_{22} A_2^T - A_2 X_{22} C_2^T \left(C_2 X_{22} C_2^T + V\right)^{-1} C_2 X_{22} A^T + B_2 B_2^T$.

4 Numerical results

We decompose the different wavefronts of the AO system over the Zernike polynomials basis of dimension $n_b = 44$. We consider a 77-actuator DM model of the `Software Package CAOS` [10], and the DM influence matrix $M_m \in \mathbf{R}^{44\times 77}$ is obtained numerically. The WFS influence matrix $M_w \in \mathbf{R}^{104\times 44}$ is also determined from a 52 subapertures Shack-Hartmann WFS model.

Atmospheric wavefronts were generated using the Software Package `CAOS` [10] developed within the homonymic `CAOS` problem-solving environment (see [12] and `http://lagrange.oca.eu/caos`). This simulation considers a 1-s temporal evolution of a 3-layer turbulent atmosphere over an 8-m telescope, and the result is a cube of 1000 consecutive propagated wavefronts (one wavefront for each temporal step of 1 ms). The main physical parameters of the simulation are: (i) a Fried parameter r_0=12 cm at λ=500 nm ; (ii) an outer scale of 25 m ; (iii) wind velocities ranging from 8 m/s to 16 m/s. The resulting 8-m wavefronts are sampled on 128×128-pixels and 2 subharmonics were added in the process of FFT-based generation of the turbulent wavefronts[13].

For a set of six temporal evolutions of turbulent wavefronts we compute, using the Burg algorithm [11], the stable system's parameters (A_0, A_1, A_2) of the AR model (3). Note that we assume that parameters (A_0, A_1, A_2) are diagonal and, as a consequence, the components of w_a are assumed uncorrelated. The associated matrix transfer function $G_a(z)$ is diagonal. We denote $G_a^{(i)}(z)$ the transfer function between input i and output i of this AR model. For each identified model, we plot the frequency response of each transfer function $G_a^{(i)}(z)$. Numerically we compute the AR model's parameters such that the associated frequency response is the mean of the set of frequency response. We call this model the *nominal* AR model. We proceed in a similar way to compute a *worst case* model which gives the higher magnitude of the frequency response. Frequency responses of transfer function $G_a^{(10)}(z)$, $G_a^{(20)}(z)$ and $G_a^{(30)}(z)$ are shown on figure 2. With the aim of satisfying performance objective (4) for the augmented system (5), the following weighting matrix of the infinite horizon quadratic cost criterion (6) is choosen as $R = r\,I$, where $r > 0$. r is a parameter to fix in order to obtain a good trade off between the minimum variance objective J_r and reasonable values of the command input. To impose a satisfactory disturbance rejection property we fix $r = 10^{-2}$. This result is numerically close to the solution of the minimum variance control problem[6]. For this cheap control case $R \to 0$, the state feedback gains are

$$K_1 = 0, \quad K_2 = \left[\, 0 \ M_m^{\dagger} A_a^2 \,\right] ,$$

where $M_m^{\dagger}$ is the Moore-Penrose pseudoinverse of the matrix M_m. The second step is the synthesis of the linear optimal state estimator gain L_2 ruled by the choice of the covariance matrix $V = 10^{-2} I$. We designed two controllers using these synthesis parameters. We call respectively nominal LQG controller (worst case LQG controller), the state space system (7) where the state feedback gains K_1, K_2 and the observer gain L_2 are computed using the parameters of the nominal AR atmospheric model (of the worst case AR atmospheric model).

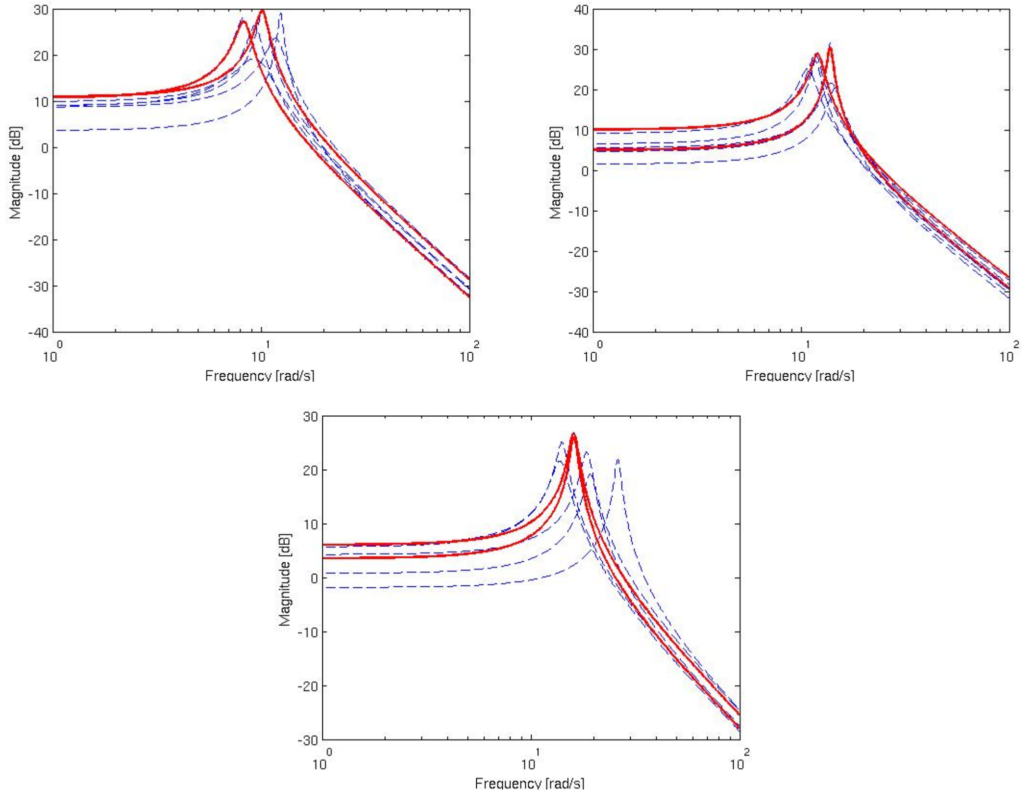

Fig. 2. Bode magnitude plots of transfert function $G_a^{(10)}(z)$, $G_a^{(20)}(z)$, $G_a^{(30)}(z)$ for the six identified model (dashed line), for the nominal AR model (plain line), and worst case AR model (plain line).

Let us now analyze the frequency response of the sensitivity transfer function, and of the disturbance rejection transfer function. Figure 3 shows the maximum singular value of $T_{11}(e^{j\omega T})$ and the maximum singular value of $T_{12}(e^{j\omega T})$ for the LQG nominal controller and for the LQG worst case controller. The difference between the two maximum singular values is small for all the frequency range. We can note that the worst case LQG controller ensures a better rejection of the atmospheric wavefront perturbation, see the singular values plot of $\bar{\sigma}(T_{11}(e^{j\omega T}))$ in figure 3(a). However the worst case LQG controller is more sensitive to the measurement noise in the high frequency range, see the singular values plot of $\bar{\sigma}(T_{12}(e^{j\omega T}))$ in figure 3(b).

The time evolution of three normalized modes of the atmospheric wavefront (sequence 1) is plotted in figure 4. The response w_r, for the AO system in closed loop with the worst case LQG controller, is also depicted in figure 4. Each peak value of the three normalized modes of the residual wavefront is reduced by a factor 20: the worst case LQG controller rejects the atmospheric wavefront perturbation. The AO loop settling time is about 150 ms.

Numerical results in terms of standard deviation are summarized in the table 1 and in the table 2. The disturbance rejection performance induced by the two controllers are acceptable for all the six simulated atmospheric wavefront sequences (as a reference, Noll residual is 278 nm for the turbulent case considered here). We can affirm that the LQG controllers ensure robust disturbance rejection performance for the given set of the atmospheric wavefront sequences. Simulation results indicate that there is no noticeable difference between the performances involved by the two controllers. In future numerical experiments, a selection of a bigger set of simulated atmospheric wavefront sequences may enlarge the set of frequency responses. This will lead to separate nominal and worst case AR models and separate LQG controllers. In this case the frequency and time response could be sensibly dissimilar and may lead to a different conclusion.

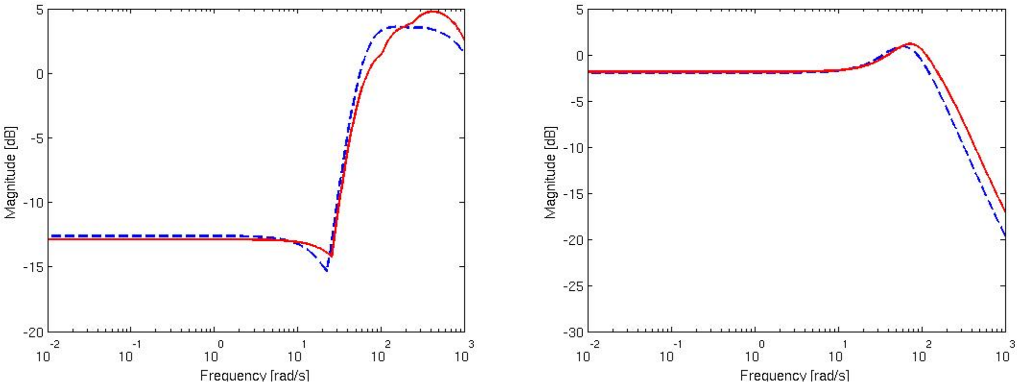

Fig. 3. Maximum sigular value of $\bar{\sigma}(T_{11}(e^{j\omega T}))$ and $\bar{\sigma}(T_{12}(e^{j\omega T}))$ for the nominal LQG controller (dashed line) and for the worst case LQG controller (plain line).

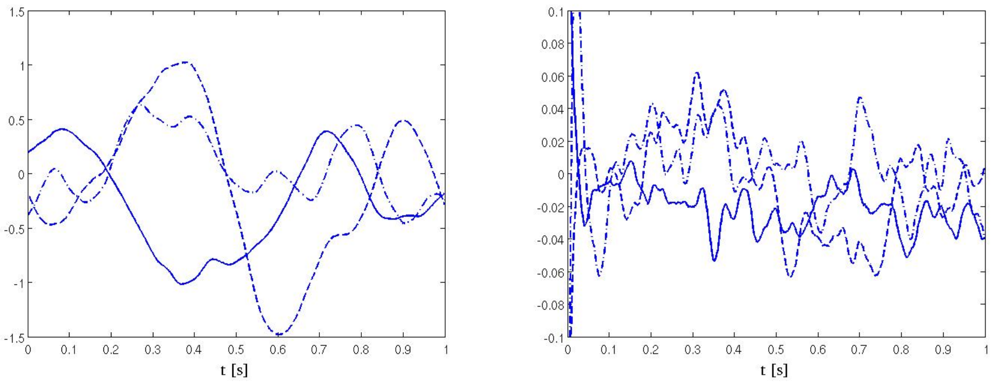

Fig. 4. w_a and w_r time evolution for the 10th mode (plain line), for the 20th mode (dashed line), and for the 30th mode (dashed-dot line).

Table 1. Standard deviation of the atmospheric wavefront sequences.

All modes standard deviation					
Sequence 1	Sequence 2	Sequence 3	Sequence 4	Sequence 5	Sequence 6
~ 1481 *nm*	~ 1280 *nm*	~ 1048 *nm*	~ 1034 *nm*	~ 1503 *nm*	~ 1190 *nm*

Table 2. Standard deviation of the residual wavefront for the two designed LQG controllers.

Controllers	All modes standard deviation					
	Sequence 1	Sequence 2	Sequence 3	Sequence 4	Sequence 5	Sequence 6
Nominal LQG controller	~ 367 *nm*	~ 310 *nm*	~ 294 *nm*	~ 294 *nm*	~ 367 *nm*	~ 325 *nm*
Worst case LQG controller	~ 365 *nm*	~ 307 *nm*	~ 291 *nm*	~ 291 *nm*	~ 362 *nm*	~ 322 *nm*

References

1. Roddier, F., *Adaptive optics in astronomy* (Cambridge Univ Pr, New York 1999).
2. Rousset, G., Lacombe, F., Puget, P., Hubin, N., Gendron, E., Fusco, T., Arsenault, R., Charton, J., Feautrier, P., Gigan, P., et al., "NAOS, the first AO system of the VLT: on-sky performance", in "Astronomical Telescope and Instrumentation, Proceedings of SPIE", **4839**, (august 2002).

3. Esposito, S., Riccardi, A., Fini, L., Puglisi, A., Pinna, E., Xompero, M., Briguglio, R., Quirós-Pacheco, F., Stefanini, P., Guerra, J., et al., "First light AO (FLAO) system for LBT: final integration, acceptance test in Europe, and preliminary on-sky commissioning results", in "Adaptive Optics Systems II, Proceedings of SPIE 7736", (july 2010).
4. Le Roux, B., Conan, J., Kulcsár, C., Raynaud, H., Mugnier, L., and Fusco, T., "Optimal control law for classical and multiconjugate adaptive optics", Journal of the Optical Society of America A **21**(7), (2004) 1261–1276.
5. Looze, D., "Minimum variance control structure for adaptative optics systems", Journal of the Optical Society of America **23**(3), (2006) 603–612 .
6. Kulcsár, C., Raynaud, H., Petit, C., Conan, J., and de Lesegno, P., "Optimal control, observers and integrators in adaptive optics", Appl. Opt **39**, (2000) 2525–2538.
7. Folcher, J., Abelli, A., Ferrari, A., and Carbillet, M., "Classic adaptive optics: disturbance rejection control", in "Adaptive Optics Systems II, Proceedings of SPIE 7736", (july 2010).
8. Hinnen, K., Verhaegen, M., and Doelman, N., "Exploiting the spatiotemporal correlation in adaptive optics using data-driven H_2-optimal control", Journal of the Optical Society of America A **24**(6), (2007) 1714–1725.
9. Skogestad, S. and Postlethwaite, I., *Multivariable Feedback Control: Analysis and Design* (John Wiley & Sons, Inc., New York 2005).
10. Carbillet, M., Vérinaud, C., Femenía, B., Riccardi, A., and Fini, L., "Modelling astronomical adaptive optics – i. the software package CAOS", in "Mon. Not. R. Astron. Soc" **356**, (2005) 1263–1275.
11. Burg, J., "Maximum entropy spectral analysis", PhD thesis, Stanford University (1975).
12. Carbillet, M., Vérinaud, C., Guarracino, M., Fini, L., Lardière, O., Le Roux, B., Puglisi, A. T., Femenía, B., Riccardi, A., Anconelli, B., Bertero, M., and Boccacci, P., "Caos - a numerical simulation tool for astronomical adaptive optics (and beyond)," in "Advancements in Adaptive Optics" **5490**, (2004) 550–559 .
13. Carbillet, M. and Riccardi, A., "Numerical modeling of atmospherically perturbed phase screens: new solutions for the classical FFT and Zernike methods," App. Opt. (OSA A/App. Opt. Joint Feature Issue on Adaptive Optics), (2010).
14. Noll, R., "Zernike polynomials and atmospheric turbulence," Journal of the Optical Society of America **66**(3), (1976) 207–211 .