ENSEMBLE TRANSFORM KALMAN FILTER : TOWARDS A DYNAMIC AND OPTIMAL CONTROL LAW FOR AO ON ELTs

M. Gray[a] and B. Le Roux

Laboratoire d'Astrophysique de Marseille - LAM (Université d'Aix Marseille & CNRS, UMR7326)
38 rue Frédéric Joliot-Curie, 13388 Marseille cedex 13, France

Abstract. Optimal control laws for new Adaptive Optics concepts (wide field tomographic AO) require the implementation of techniques intended for real time identification of the atmospheric turbulence. The Kalman Filter based control law enables estimation and prediction of the turbulent phase from the measurements and corrects efficiently the different modes of this phase on the basis used for the factorization. But using such kind of processes (within the frame of an AO system for any ELT class telescope) will be extremely difficult because of the numerical complexity of the computations involved in the matrices calculations (especially the Kalman gain which is obtained by solving the Riccati equation) as well as the recording of large covariance matrices. We will propose a new method developed for geophysics and meteorology applications which allows to dramatically reduce the computation burden while allowing to deal with dynamic (non stationary) behaviors of turbulence parameters. We briefly present a first approach: the Ensemble Kalman Filter and some drawbacks of this statistical method. Then an extension of this approach, the Ensemble Transform Kalman Filter is analyzed. A basic theoretical formulation is given and several advantages are pointed out, especially the possibility of using a non stationary model combined with distributed parallel environment implementation in the context of an ELT application. First simulations results are presented in the case of zonal Single Conjugate Adaptive Optics.

1 Introduction

On 8-10 m class telescope, classical AO systems with a feed-back loop and an optimized modal gain integrator for the control law are widespread techniques for high resolution imaging. But it has been proved that those methods are limited for tomographic AO systems designed to correct for wide Field of View (FoV). The Kalman Filter (KF) based control law allows to optimally deal with such kinds of systems (namely MCAO [1,2], LTAO or MOAO) while it is also very helpful in eXtreme AO systems to predict and compensate the telescope vibration's effects [3,4]. Nevertheless, using the KF needs to calculate the Kalman gain by solving the Riccati matrix equation. Even though this computation can be done off-line, its complexity grows very quickly with the size D_t of the telescope aperture: this computational cost is proportionnal to the third power of the number of the states of the system or to the sixth power of the diameter ($O(D_t^6)$). So it will be not possible to do it with ELTs and there is a real need for a new control law. We have proposed a new approach [5] to use the KF with an on-line computation of the Kalman gain for the AO systems in the case of ELT class telescope that enables to take into account **non stationary** modelisation of the turbulence during **all** the observation time. This new way was based on the use of Ensemble Kalman Filter (EnKF) : the idea is to compute an ensemble of **m** elements in order to implicitly represent and evolve the system's estimation error covariance matrices (for more precisely different descriptions of EnKF, you can refer to [6–8]). We have shown that the computational complexity is linear over the number **n** of degrees of freedom, and over the number **p** of degrees of observation, with a proportional factor equal to m^2 [5,9]. In this statistical method, EnKF suffers terribly from sampling error for most practical applications, owing to the insufficient ensemble size. In this paper, we will propose a semi-deterministic approach, without a randomization of data during the update step, based on the Square Root Filters analysis scheme and with the same numerical complexity: the Ensemble Transform Kalman Filter (ETKF) [10–12].

[a] morgan.gray@oamp.fr

2 Kalman Filter based control laws for the AO systems on ELTs

In the following, we are choosing a classical **zonal** Single Conjugate Adaptive Optics (SCAO) configuration in the description of the use of the KF, the EnKF and the ETKF for the estimation and the correction of the turbulent phase.

2.1 Kalman Filter (KF) & Ensemble Kalman Filter (EnKF) for AO systems on ELTs

We will use the notations and the structure of the control law presented in [2, 5, 13] with only 2 successive instants of the turbulent phase φ^{tur} in the state vector $\mathrm{X_k} = \left((\varphi_k^{tur})^T \ (\varphi_{k-1}^{tur})^T \ (u_{k-1})^T \ (u_{k-2})^T \right)^T$ of the linear state-space model:

$$\begin{aligned} \mathrm{X_{k+1}} &= \mathrm{A} \times \mathrm{X_k} + \mathrm{B} \times \mathrm{u_k} + \mathrm{V_k} \\ \mathrm{Y_k} &= \mathrm{C} \times \mathrm{X_k} + \mathrm{W_k} \end{aligned} \tag{1}$$

In the first equation, the sparse matrix A enables to characterize the evolution of the turbulence with an AR 1 or AR 2 model, the vector $\mathrm{u_k}$ is containing the tensions that are applied on the deformable mirror and the vector $\mathrm{V_k}$ is a zero mean Gaussian model noise. In the second equation, the vector $\mathrm{Y_k}$ is containing the slopes given by the Shack-Hartmann Wave Front Sensor (WFS), the sparse matrix C is the matrix of observation and $\mathrm{W_k}$ is a zero mean Gaussian measurement white noise. In the Gaussian case, the solution of the forecast estimation of $\mathrm{X_k}$ is given by a KF:

$$\hat{\mathrm{X}}_{k+1/k} = \mathrm{A_1} \times \hat{\mathrm{X}}_{k/k-1} + \mathrm{A_1} \times \mathrm{H_k} \times [\mathrm{Y_k} - \hat{\mathrm{Y}}_{k/k-1}] \tag{2}$$

with the Kalman gain:

$$\mathrm{H_k} = \Sigma_{k/k-1} \times \mathrm{C}_1^T \times [\mathrm{C_1} \times \Sigma_{k/k-1} \times \mathrm{C}_1^T + \Sigma_\mathrm{w}]^{-1} \tag{3}$$

The size of matrix $\mathrm{A_1}$ is: $n \times n$, where $n = 2\times$ number of actuators.
The size of matrix $\mathrm{C_1}$ is: $p \times n$, where $p = 2\times$ number of sub pupils.
In the use of the **KF for an AO system**, all the matrices of the state model are stationary during a part of the observation time and the asymptotic formulation of the KF is applied without loss of optimality: the Kalman gain is calculated with an off-line computation by solving the Riccati algebraic matrix equation ($\mathrm{H_k} = \mathrm{H_\infty}$). But it can take a long time (it depends on the size of the aperture) and this gain will remain the same during this part of the observation time, during which the turbulence is considered as a **stationary phenomenon**.
In the use of the **EnKF for an AO system**, we have to create a initial ensemble of m elements ($\mathrm{m} \ll \mathrm{n, p}$) from the Von Karman spatial covariance matrix and we are computing this ensemble to make a propagation of this system state. Moreover, at each step, we substitute the real covariance matrices by the covariance matrices calculated from the m elements estimates of this ensemble. For instance,
- during the forecast/prediction step [5], each ensemble's element i estimate (1≤i≤m) is computed independently according to the original state equation:

$$\hat{\mathrm{X}}^i_{k/k-1} = \mathrm{A_1} \times \hat{\mathrm{X}}^i_{k-1/k-1} + \mathrm{V}^i_\mathrm{k} \tag{4}$$

the state vector forecast estimation is:

$$\overline{x}_{k/k-1} = \frac{1}{m} \sum_{i=1}^{m} \hat{\mathrm{X}}^i_{k/k-1} \tag{5}$$

and the forecast estimation error covariance matrix is:

$$\Sigma_{k/k-1} = \frac{1}{m-1} \sum_{i=1}^{m} (\hat{\mathrm{X}}^i_{k/k-1} - \overline{x}_{k/k-1})(\hat{\mathrm{X}}^i_{k/k-1} - \overline{x}_{k/k-1})^T \tag{6}$$

- during the analysis/update step [5], each ensemble's element i estimate ($1 \leq i \leq m$) is computed independently according to the update equation:

$$\hat{X}^i_{k/k} = \hat{X}^i_{k/k-1} + H_k \times [Y_k + W^i_k - \hat{Y}^i_{k/k-1}] \tag{7}$$

and the state vector update estimation is:

$$\overline{x}_{k/k} = \frac{1}{m} \sum_{i=1}^{m} \hat{X}^i_{k/k} \tag{8}$$

Thanks to the Sherman-Morrison-Woodbury (S-M-W) formula, from (7), a new update estimate expression can be rewritten [5,9] and the inversion that appears in the Kalman gain H_k can then be calculated on a matrix with a very small size ($m \times m$).
Thus, at **each update step k**, a **new Kalman gain** H_k is obtained with the covariance matrices computed from the m elements of the ensemble.
EnKF allows bringing on AO systems all the advantages of the state-space formalism of a KF based control law with an on-line computation of the Kalman gain which then enables to have **non stationary or dynamic** models for the atmospheric turbulence **during all the observation time** and allows a reduced computational complexity in this transition to a very high number of parameters with ELT class telescope. However there is a necessity to add randomized measurement noise W^i_k and model noise V^i_k during the differents steps in order to avoid premature reduction in the spread of the ensemble's elements [6,7]. Furthermore the prediction estimate $\overline{x}_{k/k-1}$ and the update estimate $\overline{x}_{k/k}$ are statistical means (EnKF satisfies only in a statistical sense the original equations of the KF) and EnKF is a suboptimal filter for small ensembles.

2.2 Ensemble Transform Kalman Filter (ETKF) for AO systems on ELTs

ETKF is one variant of ensemble Square Root Filters (SRF): a Kalman SRF prevents the covariance matrices full estimation from the complete computation and avoids the loss of positive definiteness of those covariance matrices. As there is no need to add randomized measure noise W^i_k during the update step, ETKF is a semi-deterministic solution of EnKF. We are still using an ensemble of m elements, and we can rewrite the equations by concatenating the vectors of the m elements in one matrix whose each column is a vector $\hat{X}^i_*$ with $1 \leq i \leq m$.
Thus, for the analysis/update step (k/k) and for the forecast/prediction step (k+1/k), we have to define:
- a matrix of elements estimates:

$$\hat{X}_* = [\hat{X}^1_* \, ; \, ... \, ; \, \hat{X}^m_*] \tag{9}$$

- a matrix of anomalies:

$$Z_* = \frac{[z^1_* \, ; \, ... \, ; \, z^m_*]}{\sqrt{m-1}} \quad \text{with} \quad z^i_{k/k} = \hat{X}^i_{k/k} - \hat{x}_{k/k} \quad \text{or} \quad z^i_{k+1/k} = \hat{X}^i_{k+1/k} - \overline{x}_{k+1/k} \tag{10}$$

- a covariance matrix:

$$\Sigma_* = (Z_*) \times (Z_*)^T \quad \text{that is not computed !} \tag{11}$$

Only the matrices of anomalies are calculated and the creation of the anomalies update matrix from the anomalies prediction matrix is obtained by a multiplication with the matrix of transformation T_k.
We can sum up the different steps of this new method:
With the m initial elements obtained from the Von Karman spatial covariance matrix, we compute an initial prediction (of the state-vectors matrix X_1): $\hat{X}_{1/0}$
and an inital anomalies prediction matrix: $Z_{1/0}$
Each following step ($k \geq 1$) can be divided in **two stages**.
During the update stage, we have to compute:
- an Eigen Values Decomposition (EVD) of a matrix of size (m×m) and the matrix of transformation:

$$T_k = [I_m + (C_1 Z_{k/k-1})^T \times \Sigma_w^{-1} \times (C_1 Z_{k/k-1})]^{-\frac{1}{2}} \tag{12}$$

- the Kalman gain is not directly computed with the expression (3). As it was done with EnKF, by using the previous EVD and the S-M-W formula, the update estimate is rewritten [5,9] from:

$$\hat{x}_{k/k} = \overline{x}_{k/k-1} + \mathrm{H_k} \times [\mathrm{Y_k} - \hat{\mathrm{Y}}_{k/k-1}] \quad (13)$$

- the anomalies update matrix:

$$\mathrm{Z}_{k/k} = \mathrm{Z}_{k/k-1} \times \mathrm{T}_k \quad (14)$$

- the elements estimates update matrix:

$$\hat{\mathrm{X}}_{k/k} = \sqrt{m-1} \times \mathrm{Z}_{k/k} + [\hat{\mathrm{x}}_{k/k}\,;\,...\,;\,\hat{\mathrm{x}}_{k/k}] \quad (15)$$

During the prediction stage, we have to compute:
- the prediction of each ensemble's element estimate: $\hat{\mathrm{X}}^i_{k+1/k}$ (1≤i≤m) with (4) (as EnKF).
- the prediction estimate: $\overline{x}_{k+1/k}$ with (5) (as EnKF).
- the anomalies prediction matrix:

$$\mathrm{Z}_{k+1/k} = [\hat{\mathrm{X}}^1_{k+1/k} - \overline{\mathrm{x}}_{k+1/k}\,;\,...\,;\,\hat{\mathrm{X}}^m_{k+1/k} - \overline{\mathrm{x}}_{k+1/k}]/\sqrt{m-1} \quad (16)$$

With this forecast estimate $\overline{\mathrm{x}}_{k+1/k}$ computed **during the step k**, we can have an estimate $\hat{\varphi}^{tur}_{k+1/k}$ of the turbulent phase (on the pupil of the telescope) of the instant k+1, and then apply the right correction on the deformable mirror of the AO system during this **next step k+1**.

2.3 Advantages of ETKF

The total numerical complexity of ETKF is **still linear over n and over p**: $O(m^2 \times (n+p))$
The Eigen Values Decomposition and the inversion are computed on small matrices with a size of m×m. As written before, the full estimation error covariance matrices are not computed (only the square root matrices Z_*) which preserves the positive definiteness of those covariance matrices.
The Kalman gain can be calculated at each step during the observation and therefore ETKF is **well suited for non stationary or dynamic** state-space model of the turbulence.
The structure of ETKF and the implementation is well adapted with a **distributed parallel environment** because this method is naturally parallel (it will be done by the CeSAM team at the LAM).
Finally, ETKF could be adapted for **non linear** state-space models which can be very useful for an AO system with a pyramid WFS, for taking into account the saturation of the actuators during strong turbulence or, perhaps, for a real time estimation of the turbulent parameters.

3 Very first results of numerical simulations (Zonal SCAO)

3.1 Simulation conditions

For the atmosphere, we consider a Von Karman turbulence: $r_0 = 0.525$ m, $L_0 = 25$ m, $\lambda = 1.654\,\mu$m (for both WFS's *and* observation's wavelengths). Using taylor's hypothesis under OOMAO Matlab environnement, we can generate a superimposition of 3 turbulent phase screen layers moving at 7.5 ms^{-1}, 12.5 ms^{-1}, 15 ms^{-1} with a relative strength of 0.5, 0.17, 0.33 respectively.
For this first AO system simulation, we consider one telescope with a diameter of 4.2 m, with 2 configurations: a 8×8 *or* a 20×20 microlenses Shack-Hartmann WFS (giving the slopes of the turbulent phase on each subpupil). Both system works in a close loop at 250 Hz and there is two-step delay between measurement and correction. We assume that the DM has an instantaneous response and the coupling factor of the actuators is 0.3. Phase screens are generated on a 80×80 grid (or 200×200 grid), with 10 times 10 points per each subaperture and the phase is estimated only on the actuators' locations (the number of valid actuators is 69 for the 8×8 WFS and 357 for the 20×20 WFS).
For both model used by the KF *and* by the ETKF, we have chosen a first order AR model and the number of the ensemble's elements is 69 (for the 8×8 WFS) and 300 (for the 20×20 WFS).
Each value of the Strehl ratio has been calculated with 2 simulations of 3000 iterations (12 sec).

3.2 Simulation results

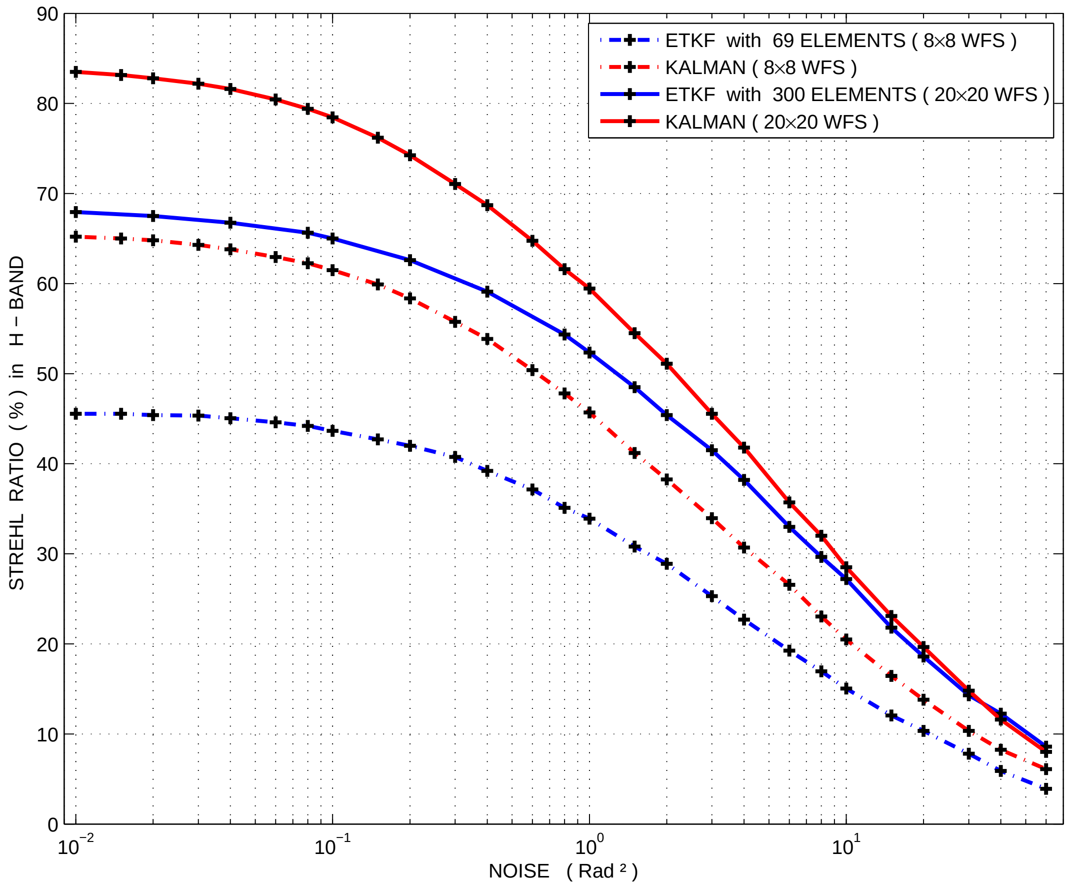

Fig. 1. Strehl ratio for 2 ZONAL SCAO configurations (8×8 or 20×20 WFS)

4 Conclusions and perspectives

If we compare the results in both AO configurations, we can notice of course that the performances obtained with ETKF are not so good as those obtained with the KF. Moreover, the differences between the 2 Strehl ratios are bigger when the physical noise is weak which could appear to be quite strange. When the number of sub apertures is increasing (we can increase the number of elements), this difference is decreasing a little for a given physical noise. But these simulation results were calculated from a very first and basic implementation of this new control law: with ETKF, there are different parameters we can change and we do not have to forget that this new control law offers non stationary modelisation of the turbulence. It was the first step to prove that ETKF based control law can be implemented on AO sytems for astronomy. For the following steps, we will study different theoretical points and different configurations of the AO parameters in order to understand how to improve this use of ETKF (an article about ETKF for AO systems on ELTs is in preparation and will review the complete theoretical formulation, a computational complexity discussion, the feasable parallel implementation and different numerical simulations results).

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